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SOFIC SYSTEMS

BY

ETHAN M. COVEN AND MICHAEL E. PAUL

ABSTRACT

A symbolic flow is called a sofic system if it is a homomorphic image (factor) of a subshift of finite type. We show that every sofic system can be realized as a finite-to-one factor of a subshift of finite type with the same entropy. From this it follows that sofic systems share many properties with subshifts of finite type. We concentrate especially on the properties of TPPD (transitive with periodic points dense) sofic systems.

Introduction

Sofic systems, the class of symbolic flows which are factors of subshifts of finite type, were introduced by Weiss in [15]. Weiss found sufficient, but not necessary, conditions for a sofic system to be intrinsically ergodic, i.e., to possess a unique entropy-maximizing measure. This paper is an outgrowth of the authors' search for necessary and sufficient conditions for intrinsic ergodicity.

In Section 1 we discuss some basic facts about subshifts of finite type and sofic systems. In Section 2 we show that every sofic system may be realized as a finite-to-one factor of a subshift of finite type with the same topological entropy. In Section 3 we show that a sofic system is transitive with periodic points dense (TPPD) if and only if it is intrinsically ergodic with support (IES). (A flow is IES if it is intrinsically ergodic and the unique entropy-maximizing measure is positive on non-empty, open sets.) In Section 4 we examine some properties of TPPD sofic systems. In Section 5 we raise the question of how to distinguish (dynamically as opposed to combinatorially) between subshifts of finite type and "strictly" sofic systems.

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1. Subshifts of finite type and sofic systems

Let $(X(m),\sigma)$ denote the full shift system on m symbols;

$$X(m)=\{1,2,\cdots,m\}^Z$$
$$=\{x=(x_i)\,|\,x_i=1,2,\cdots,m\,;i=0,\pm 1,\pm 2,\cdots\}$$

and the shift homeomorphism σ is defined by $[\sigma(x)]_i=x_{i+1}$. The symbol set $\{1,2,\cdots,m\}$ may be replaced by any other set of cardinality m.

A *symbolic flow* (X,σ) is a subflow of some full shift system. An n-block $B=b_1\cdots b_n$ *appears in* X if $B=x_i\cdots x_{i+n-1}$ for some $x\in X$ and some integer i. Let

$$\mathcal{B}(X,n)=\{B\,|\,B \text{ is an } n\text{-block}, B \text{ appears in } X\}.$$

An *n-block map* of (X,σ) is a map $f:\mathcal{B}(X,n)\to\{1,2,\cdots,m\}=\mathcal{B}(X(m),1)$ for some $m\geqq 1$. An n-block map f gives rise to a map of $\mathcal{B}(X,n+1)$ into $\mathcal{B}(X(m),2)$, also denoted by f, defined by $f(b_1\cdots b_{n+1})=f(b_1\cdots b_n)f(b_2\cdots b_{n+1})$. Similarly, for each $k\geqq 1$, f maps $\mathcal{B}(X,n+k-1)$ into $\mathcal{B}(X(m),k)$. Define $f_\infty:X\to X(m)$ by $f_\infty(x)=y$ where $y_i=f(x_i\cdots x_{i+n-1})$. Then f_∞ is a homomorphism of (X,σ) into $(X(m),\sigma)$.

THEOREM 1.1. (Curtis-Hedlund-Lyndon) *Let* $\pi:(X,\sigma)\to(Y,\sigma)$ *be a homomorphism of one symbolic flow to another. Then there exist integers* $n\geqq 1$ *and* k *and an n-block map* f *of* (X,σ) *such that* $\pi=\sigma^k f_\infty$.

PROOF. The proof of [10, Theorem 3.4] for $X=Y=X(m)$ is valid in this case as well.

Let M be an $m\times m$ matrix of zeros and ones and let

$$X(M)=\{x\in X(m)\,|\,M(x_i,x_{i+1})=1 \text{ for all } i\}.$$

Then $X(M)$ is a closed, invariant subset of $X(m)$. A symbolic flow is called a *subshift of finite type* if it is isomorphic to some $(X(M),\sigma)$, where of course $X(M)\neq\varnothing$. Just as $X(M)$ is "determined by its 2-blocks", (X,σ) is a subshift of finite type if and only if X is "determined by its n-blocks" for some $n\geqq 1$ (cf. the original definitions of Parry [12] and Bowen [2]).

Throughout this paper we shall assume that M *has no row or column which consists only of zeros.* Then $X(M)\neq\varnothing$ and every M-admissible block appears in $X(M)$. (A block $B=b_1\cdots b_n$ is M-admissible if $M(b_1b_2)\cdots M(b_{n-1}b_n)=1$.) No loss of generality results from this assumption.

We will sometimes find it useful to assume that M is in normal form (see Gantmacher [8, p. 75]). By reordering the indices of M (which corresponds to an isomorphism of $(X(M),\sigma)$ via a permutation of the symbol set) we may write

$$M = \begin{pmatrix} M_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ * & & & M_r \end{pmatrix}$$

where the diagonal blocks $M_1, \cdots, M_r$ are square and each M_k is either irreducible or the one-by-one zero matrix. It is easily vertified that $X(M) \neq \emptyset$ if and only if at least one M_k is not the zero matrix.

Let M_k be indexed by $I_k = \{i_{k-1}+1, \cdots, i_k\}$, where $i_0 = 0$. If $M_k \neq 0$, we shall consider $(X(M_k),\sigma)$ to be a subshift of finite type with symbol set I_k.

A symbolic flow is called a *sofic system* if it is a factor (i.e., homomorphic image) of a subshift of finite type. This is equivalent to Weiss' definition by Theorem 1.1 and [15, Theorems 2 and 3].

THEOREM 1.2. *Every sofic system (Y,σ) may be realized by a 2-block map of a subshift of finite type $(X(M),\sigma)$.*

PROOF. There exists a subshift of finite type (X,σ) and an onto homomorphism $\pi: (X,\sigma) \to (Y,\sigma)$. Without loss of generality, $X = X(N)$ for some $0-1$ matrix N and $\pi = g_\infty$ where g is a k-block map of $(X(N),\sigma)$ for some $k \geqq 2$. Let M be the matrix obtained by coding the $(k-1)$-blocks of $X(N)$; if $\mathcal{B}(X(N), k-1) = \{B_1, \cdots, B_m\}$, then M is the $m \times m$ matrix defined by $M(ij) = 1$ if and only if $b_1^{(i)} B_j = B_i b_{k-1}^{(j)}$ and this k-block appears in $X(N)$. It is easily vertified that no row or column of M consists only of zeros. Let h be the $(k-1)$-block map of $(X(N),\sigma)$ defined by $h(B_i) = i$. Then h_∞ is an isomorphism of $(X(N),\sigma)$ onto $(X(M),\sigma)$. Define f by $f(ij) = g(b_1^{(i)} B_j)$. Then f is a 2-block map of $(X(M),\sigma)$, and $f_\infty[X(M)] = Y$.

2. 2-block maps of subshifts of finite type

A *partition* of a $0-1$ matrix M is an ordered collection of $0-1$ matrices $(M_1, \cdots, M_p)$ such such that $M = M_1 + \cdots + M_p$. It will always be clear from the context whether M_k denotes a diagonal block in the normal form of M or a member of a partition of M.

A partition of M induces a 2-block map f of $(X(M),\sigma)$ by $f(ij) = k$ if and only if $M_k(ij) = 1$. Conversely, a 2-block map of $(X(M),\sigma)$ induces a partition of M. The correspondence between partitions and 2-block maps is one-to-one.

For a 2-block map f of $(X(M),\sigma)$ we denote the image set $f_\infty[X(M)]$ by $X(f)$. It follows that

$$X(f)=\{x\in X(p)\,|\,M_{x_i}\cdots M_{x_j}\neq 0 \text{ for all } i\leq j\},$$

where $(M_1,\cdots,M_p)$ is the corresponding partition of M (cf. Weiss' definition of sofic system [15, p. 463]). By Theorem 1.2 the class of sofic systems is precisely the class of all $(X(f),\sigma)$ where f is a 2-block map of some $(X(M),\sigma)$.

Let $\mathcal{M}(f)$ denote the matrix semigroup generated by $\{M_1,\cdots,M_p\}$. The semigroup $\mathcal{M}(f)$ reflects certain properties of the maps f and f_∞, as will be made explicit in Lemma 2.2 and Theorem 2.3.

Before proceeding further, we show how our way of looking at sofic systems fits into Weiss' way. We assume familiarity with the notation of [15].

THEOREM 2.1. *Let $(X(f),\sigma)$ be a sofic system. Then there exists a finite semigroup G such that $X(f)=X_G$.*

PROOF. Let $\{0,*\}$ be the trivial Boolean algebra; i.e., $*+*=*$. For $W\in\mathcal{M}(f)$, let W^* be the matrix over $\{0,*\}$ obtained by replacing the non-zero entries of W by stars. Then $(W_1W_2)^*=W_1^*W_2^*$ and $M_{i_1}\cdots M_{i_n}=0$ if and only if $M^*_{i_1}\cdots M^*_{i_n}=0$. The semigroup $\mathcal{M}^*(f)$ generated by $\{M^*_1,\cdots,M^*_p\}$ is finite and $X(f)=X_{\mathcal{M}^*(f)}$.

REMARK. If $\mathcal{M}(f)$ is finite, then $X(f)=X_{\mathcal{M}(f)}$.

LEMMA 2.2. (cf. [9, Theorems 8.14 and 8.28]) *The semigroup $\mathcal{M}(f)$ is finite if and only if $\{\text{card}\, f^{-1}(B)\,|\,B$ appears in $X(f)\}$ is bounded.*

PROOF. As easy induction argument shows that if $W=M_{i_1}\cdots M_{i_n}\in\mathcal{M}(f)$, then $W(ij)=\text{card}\,\{B\in f^{-1}(i_1\cdots i_n)\,|\,B$ begins with i and ends with $j\}$. Thus $\{\text{card}\; f^{-1}(B)\,|\,B$ appears in $X(f)\}=\{\Sigma_{ij}W(ij)\,|\,W\in\mathcal{M}(f)\}$.

If $\mathcal{M}(f)$ is finite, then this set is bounded.

If the set is bounded, then $\mathcal{M}(f)$ is finite because there are only finitely many matrices of a given order with non-negative integer entries whose sum is less than a given integer.

THEOREM 2.3. *Let $\{\text{card}\, f^{-1}(B)\,|\,B$ appears in $X(f)\}$ be bounded. Then*

(1) $h(X(M),\sigma)=h(X(f),\sigma)$, where h denotes topological entropy.

(2) f_∞ is uniformly finite-to-one; i.e., $\{\text{card}\, f_\infty^{-1}(y)\,|\,y\in X(f)\}$ is bounded.

PROOF. (1) It is well-known that for (X,σ), a symbolic flow $h(X,\sigma)=\lim_{n\to\infty}(1/n)\log P(X,n)$ where $P(X,n)=\text{card}\,\mathcal{B}(X,n)$. If B is an n-block appearing in $X(f)$, then $f^{-1}(B)$ is a collection of $(n+1)$-blocks appearing in $X(M)$. Suppose that card $f^{-1}(B) \leqq K$ for all blocks B appearing in $X(f)$. Then

$$\begin{aligned} h(X(M),\sigma) &= \lim_{n\to\infty} \frac{1}{n} \log P(X(M),n) \\ &= \lim_{n\to\infty} \frac{1}{n+1} \log P(X(M),n+1) \\ &\leqq \lim_{n\to\infty} \frac{1}{n+1} \log [K \cdot P(X(f),n)] \\ &= \lim_{n\to\infty} \frac{1}{n} \log P(X(f),n) \\ &= h(X(f),\sigma) \\ &\leqq h(X(M),\sigma). \end{aligned}$$

Therefore $h(X(M),\sigma) = h(X(f),\sigma)$.

(2) If card $f^{-1}(B) \leqq K$ for all blocks B appearing in $X(f)$, then card $f_\infty^{-1}(y) \leqq K$ for all $y \in X(f)$.

REMARK. We shall show later (Theorem 4.3) that if M is irreducible, then each of the conditions (1) and (2) implies that $\{\text{card} f^{-1}(B)\}$ is bounded. The reader will recognize that (2) $\Rightarrow$ (1) is a special case of Bowen's "finite-to-one maps preserve entropy" theorem (see [4, Theorem 17]). We choose not to use this fact in order to stay within the framework of symbolic dynamics.

THEOREM 2.4. *Let (Y,σ) be a sofic system. Then there is a subshift of finite type $(X(M),\sigma)$ and a 2-block map f of $(X(M),\sigma)$ such that $Y = X(f)$ and $\mathcal{M}(f)$ is finite.*

PROOF. We may assume that $Y = X(g)$ for a 2-block map of some $(X(N),\sigma)$. Let $(N_1,\cdots,N_p)$ be the corresponding partition of N and form the semigroup $\mathcal{M}^*(g)$ as in the proof of Theorem 2.1. Define matrices $A_1,\cdots,A_p$ indexed by $(\mathcal{M}^*(g)-\{0\})\times\{1,\cdots,p\}$ as follows: $A_k[(R,i)(S,j)] = 1$ if $i = k$ and $R = N_k^* S$, and $A_k[(R,i)(S,j)] = 0$ otherwise. Then each A_k is a $0-1$ matrix and so is $A = A_1 + \cdots + A_p$.

The matrix A may contain a row or column which consists only of zeros. If the ith row or ith column of A consists only of zeros, form a new matrix by deleting both the ith row and the ith column of A. Repeated application of this procedure will produce a matrix M with no row or column consisting only of zeros such that, with an appropriate indexing set for $M, X(M) = X(A)$. If the same rows and columns that were removed from A to form M are removed from each A_k, then we obtain $0-1$ matrices $M_1, \cdots, M_p$ such that $M = M_1 + \cdots + M_p$. It is clear that if f is the 2-block map of $(X(M), \sigma)$ corresponding to $(M_1, \cdots, M_p)$ then $Y = X(f)$.

The matrices $A_1, \cdots, A_p$ have the property that their column sums are all either 0 or 1. Therefore $M_1, \cdots, M_p$ have the same property. Since this property is closed under matrix products, every matrix in $\mathcal{M}(f)$ also has this property. It follows that $\mathcal{M}(f)$ is finite.

REMARK. The matrix M constructed in the proof is essentially the transpose of the matrix T which Weiss uses to prove that a sofic system is a factor of a subshift of finite type [15, Theorem 3].

COROLLARY 2.5. *Every sofic system can be realized as a factor of a subshift of finite type with the same entropy.*

COROLLARY 2.6. *The entropy of a sofic system is the logarithm of an algebraic number.*

PROOF. It is well-known that $h(X(M), \sigma) = \log \beta$ where β is the largest eigenvalue of M.

3. TPPD = IES for sofic systems

A flow is called TPPD if it is transitive (i.e., contains a dense orbit) and the periodic points are dense. It is well-known that a subshift of finite type is TPPD if and only if it is isomorphic to some $(X(M), \sigma)$ where M is irreducible.

A flow (X, T) is called *intrinsically ergodic* if there is a unique invariant (necessarily ergodic) measure μ on X such that $h_\mu(X, T) = h(X, T) < \infty$. If, furthermore, $\mu(U) > 0$ for each non-empty, open subset U of X, then (X, T) is called IES (intrinsically ergodic with support). From the standpoint of topological dynamics it is somewhat more natural to study IES flows rather than intrinsically ergodic flows. This is because dynamical conclusions about an intrinsically ergodic flow (X, T) are essentially restricted to the IES subflow (X', T) where X' is the support of μ. Indeed, all of the examples of intrinsically ergodic flows given by Weiss in [14] are IES.

THEOREM 3.1. *A subshift of finite type* (X, σ) *is* TPPD *if and only if it is* IES.

PROOF. If (X,σ) is TPPD, then we may assume that $X = X(M)$ for some irreducible $0-1$ matrix M. Then [1, Theorem 4.1] $h(X,\sigma) < \infty$ and there is a unique entropy-maximizing measure. An examination of this measure (see [1, p. 13] or [7, Section 4]) shows that it is positive on open sets.

Suppose that (X,σ) is IES. Since any invariant measure is concentrated on the non-wandering set and the non-wandering set is closed, (X,σ) is pointwise non-wandering. Without loss of generality, $X = X(M)$ where M is in normal form:

$$M = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_r \end{pmatrix}$$

the zero in the lower left occurring because (X,σ) is pointwise non-wandering. But $\{X(M_k)\}$ is a pairwise disjoint collection of closed, invariant sets and the unique entropy-maximizing measure μ is ergodic, so μ must be concentrated on one of the $X(M_k)$. Since μ is positive on non-empty, open sets, there is exactly one non-empty $X(M_k)$. Thus M is irreducible. Hence (X,σ) is TPPD.

LEMMA 3.2. *Every subshift of finite type $(X(M),\sigma)$ contains a* TPPD *subshift of finite type $(X(M'),\sigma)$ such that $h(X(M'),\sigma) = h(X(M),\sigma)$.*

PROOF. Recall that $h(X(M),\sigma) = \log \beta$ where β is the largest eigenvalue of M. If M is in normal form, then $\beta = \max\{\beta_1,\cdots,\beta_r\}$ where β_k is the largest eigenvalue of M_k. Thus $h(X(M),\sigma) = h(X(M_k),\sigma)$ for some k and $(X(M_k),\sigma)$ is TPPD.

THEOREM 3.3. *Every sofic system (Y,σ) contains a* TPPD *sofic system (Y',σ) such that $h(Y',\sigma) = h(Y,\sigma)$.*

PROOF. Let f and M be as in Theorem 2.4 and M' as in Lemma 3.2. Let $f' = f \mid \mathcal{B}(X(M'),2)$. Then $\{\text{card } f'^{-1}(B) \mid B \text{ appears in } X(f')\}$ is bounded and so $h(Y,\sigma) = h(X(M),\sigma) = h(X(M'),\sigma) = h(X(f'),\sigma)$. Since any factor of a TPPD flow is TPPD, $(X(f'),\sigma)$ is TPPD.

THEOREM 3.4. *Every* TPPD *sofic system (Y,σ) is a factor of a* TPPD *subshift of finite type (X,σ) such that $h(X,\sigma) = h(Y,\sigma)$.*

PROOF. Again let f and M be as in Theorem 2.4. Let $y \in Y$ be a transitive point and let $x \in f_\infty^{-1}(y)$.

Suppose y is positively transitive; we show that $f_\infty[\omega(x)] = Y$. Let $q \in Y$ be a periodic point. There exist $n_i \to \infty$ such that $\sigma^{n_i}(y) \to q$. Then for any limit point p of $\{\sigma^{n_i}(x)\}, p \in \omega(x)$ and $f_\infty(p) = q$. Thus the closed set $f_\infty[\omega(x)]$ contains a dense subset of Y, hence $f_\infty[\omega(x)] = Y$.

If M is in normal form, then $\omega(x) \subseteq X(M_k)$ where k is the unique integer satisfying $\min\{x_i\} \in I_k$. Thus f_∞ maps the TPPD subshift of finite type $(X(M_k),\sigma)$ onto (Y,σ). Then $h(Y,\sigma) \leqq h(X(M_k),\sigma) \leqq h(X(M),\sigma) = h(Y,\sigma)$ and so $h(X(M_k),\sigma) = h(Y,\sigma)$.

The proof if y is negatively transitive is analogous.

Remark. In Weiss' terminology, for a sofic system (X_G,σ) where $gG \neq 0$ and $Gg \neq 0$ for all $g \neq 0$, we have (X_G,σ) is TPPD if and only if $gGh \neq 0$ for all $g, h \neq 0$.

Theorem 3.5. *A sofic system (Y,σ) is* TPPD *if and only if it is* IES.

Proof. If (Y,σ) is TPPD, then by Theorem 3.4, it is a factor of a TPPD subshift of finite type (X,σ) with the same entropy. But (X,σ) is IES and any entropy-preserving factor of an IES flow is IES.

Let (Y,σ) be IES. By Theorem 3.3, (Y,σ) contains a TPPD sofic system (Y',σ) such that $h(Y',\sigma) = h(Y,\sigma)$. But an IES symbolic flow has no proper subflow with the same entropy [7, Theorem 3.3]. Therefore $Y' = Y$.

Corollary 3.6. *Let (Y,σ) be a sofic system. Then the following statements are equivalent.*

(1) *(Y,σ) is* TPPD.
(2) *(Y,σ) is* IES.
(3) *(Y,σ) is a factor of a* TPPD *subshift of finite type with the same entropy.*
(4) *(Y,σ) is a factor of a* TPPD *subshift of finite type.*

4. TPPD sofic systems

In [15, Lemma 2], Weiss shows that a symbolic flow is intrinsically ergodic if there is an ergodic, entropy-maximizing measure (which of course turns out to be unique) that satisfies a one-sided inequality for blocks.

Theorem 4.1. *Let (Y,σ) be a* TPPD *sofic system with unique entropy-maximizing measure ν. Then there exist constants $w, W > 0$ such that for every n-block B appearing in Y, $w/\beta^n \leqq \nu(B) \leqq W/\beta^n$ where $h(Y,\sigma) = \log \beta$.*

Proof. There exist an irreducible $0-1$ matrix M, an integer $K > 0$ and a 2-block map f of $(X(M),\sigma)$ such that $Y = X(f)$ and card $f^{-1}(B) \leqq K$ for all blocks B appearing in Y. If $h(X(M),\sigma)=\log\beta$, then by [1, p. 13] or [7, Theorem 5.2], if μ is the unique entropy-maximizing measure of $(X(M),\sigma)$, there are constants $v, V>0$ such that for every n-block B appearing in $X(M), v/\beta^n \leqq \mu(B) \leqq V/\beta^n$. Furthermore, $\nu(E)=\mu[f_\infty^{-1}(E)]$ for every Borel set E of Y.

Let B be an n-block appearing in Y. Then $f^{-1}(B)$ is a collection of $(n+1)$-blocks appearing in $X(M)$, $1 \leqq \operatorname{card} f^{-1}(B) \leqq K$ and $\nu(B)=\mu[f^{-1}(B)]$. Therefore $v/\beta^{n+1} \leqq \nu(B) \leqq KV/\beta^{n+1}$.

THEOREM 4.2. *A sofic system (Y,σ) is* IES *if and only if there is an ergodic measure ν on Y, constants $w, W>0$ and $\beta \geqq 1$ such that $h(Y,\sigma)=h_\nu(Y,\sigma)$ and $w/\beta^n \leqq \nu(B) \leqq W/\beta^n$ for all n-blocks B appearing in Y.*

PROOF. If the conditions are satisfied, then $h(Y,\sigma)=\log\beta$ and by [15, Lemma 2] ν is the unique entropy-maximizing measure of (Y,σ). Since ν is positive on open sets, (Y,σ) is IES.

THEOREM 4.3. *Let f be a 2-block map of $(X(M),\sigma)$ where M is irreducible. Then the following statements are equivalent.*

(1) *$\mathcal{M}(f)$ is finite.*

(2) *$\{\operatorname{card} f^{-1}(B) \mid B$ appears in $X(f)\}$ is bounded.*

(3) *$h(X(M),\sigma)=h(X(f),\sigma)$.*

(4) *f_∞ is finite-to-one.*

(5) *f_∞ is uniformly finite-to-one.*

PROOF. By Lemma 2.2 and Theorem 2.3, it suffices to show (3)$\Rightarrow$(2) and (4)$\Rightarrow$(2).

(3)$\Rightarrow$(2): suppose $h(X(M),\sigma)=h(X(f),\sigma)=\log\beta$. Let μ and ν be the unique entropy-maximizing measures of $(X(M),\sigma)$ and $(X(f),\sigma)$ with constants v and W as described in Theorems 4.1 and 4.2. Let B be an n-block appearing in $X(f)$. Then $f^{-1}(B)$ is a collection $(n+1)$-blocks appearing in $X(M)$ and

$$\frac{[\operatorname{card} f^{-1}(B)]v}{\beta^{n+1}} \leqq \mu[f^{-1}(B)] = \nu(B) \leqq \frac{W}{\beta^n}.$$

Therefore $\operatorname{card} f^{-1}(B) \leqq W\beta/v$.

(4)$\Rightarrow$(2): if $\{\operatorname{card} f^{-1}(B) \mid B$ appears in $X(f)\}$ is unbounded, then there exists $W \in \mathcal{M}(f)$ with some entry $W(ij) \geq 2$. Hence there are distinct blocks B and B' of the same length, both appearing in $X(M)$, such that $f(iBj)=f(iB'j)$. Since M is irreducible, there is a block of the form jCi which appears in $X(M)$.

Therefore $A = iBjC$ and $A' = iB'jC$ are distinct blocks of the same length, both of which appear in $X(M)$ and are mapped to the same block by f. Then the set $X = \{\cdots D_{-1}D_0D_1\cdots \mid D_k = A \text{ or } A'\}$ is an infinite, indeed uncountable, subset of $X(M)$ which is mapped by f_∞ to a single point.

REMARK. In view of [7, Theorem 3.3] (an IES symbolic flow contains no proper subflow with the same entropy), Theorem 4.3 may be thought of as an extension of Theorems 5.6 and 5.12 of [10] which state that an endomorphism of $(X(n),\sigma)$ is onto if and only if it is uniformly finite-to-one; see also [7, Theorems 5.7 and 5.8].

REMARK. We cannot drop irreducibility from the statement of Theorem 4.3. Let

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

so that $X(M)$ is two copies of the full 2-shift, and let f correspond to the partition

$$\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right).$$

Then f_∞ is infinite-to-one (it is the identity on one copy and maps the other copy to a single point) but $X(f)$ is the full 2-shift.

COROLLARY 4.4. *Let π: $(X,\sigma)\to(Y,\sigma)$ be a homomorphism of a* TPPD *subshift of finite type onto a sofic system. Then the following statements are equivalent.*

(1) *π is finite-to-one.*

(2) *π is uniformly finite-to-one.*

(3) $h(X,\sigma) = h(Y,\sigma)$.

COROLLARY 4.5. *Let π: $(Y_1,\sigma)\to(Y_2,\sigma)$ be a homomorphism of one* TPPD *sofic system onto another. Then the following statements are equivalent.*

(1) *π is finite-to-one.*

(2) *π is uniformly finite-to-one.*

(3) $h(Y_1,\sigma) = h(Y_2,\sigma)$.

PROOF. There is a uniformly finite-to-one homomorphism ϕ of a TPPD subshift of finite type (X, σ) onto (Y_1, σ). Apply Corollary 4.4 to the map $\pi\phi$.

It is now easy to extend the results on inverses of onto endomorphisms in [10, sections 9–12] and [7, section 6] to finite-to-one homomorphisms between TPPD sofic systems. Bowen ([4] and [5, section 4]) has similar results for homomorphisms of subshifts of finite type to Axiom A diffeomorphisms.

The next theorem gives a finite procedure for deciding whether the conditions of Theorem 4.3 are satisfied. It may be thought of as an extension of Blackwell's Theorem [9, Theorem 8.7] which states that an n-block map of the full 2-shift defines an onto endomorphism if card $f^{-1}(B) = 2^{n-1}$ for all blocks B of length less than or equal to 2^{n-1}.

THEOREM 4.6. *Let M be an irreducible $m \times m$ matrix of zeros and ones and let f be a 2-block map of $(X(M), \sigma)$ with corresponding partition $(M_1, \cdots, M_p)$. Then the conditions of Theorem 4.3 are satisfied if and only if every product of the form $M_{i_1} \cdots M_{i_k}$ with $1 \leqq k \leqq 2^{m^2} + 1$ is a $0-1$ matrix.*

PROOF. The proof of (4)$\Rightarrow$(2) of Theorem 4.3 shows that (4) implies that every member of $\mathcal{M}(f)$ is a $0-1$ matrix.

Suppose every k-fold product of $M_i s$ is a $0-1$ matrix for $1 \leqq k \leqq 2^{m/2} + 1$. Define subsets $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ of $\mathcal{M}(f)$ by $\mathcal{Y}_n = \{W \in \mathcal{M}(f) \mid W = M_{i_1} \cdots M_{i_k}$ for some $k \leqq n\}$. Then each $\mathcal{Y}_n$ is finite, $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$ and if $\mathcal{Y}_n = \mathcal{Y}_{n+1}$, then $\mathcal{Y}_n = \mathcal{Y}_{n+1} = \mathcal{Y}_{n+2} = \cdots = \mathcal{M}(f)$. Since there are only 2^{m^2} matrices of order m with entries 0 or 1, we must have $\mathcal{Y}_n = \mathcal{Y}_{n+1}$ for some $n \leqq 2^{m^2} + 1$.

REMARK. The bound $2^{m^2} + 1$ given in the theorem is extremely crude. In our setting, Blackwell's bound would be m.

5. Strictly sofic systems

A sofic system which is not a subshift of finite type is called *strictly sofic*. Are there dynamical properties which distinguish between subshifts of finite type and strictly sofic systems? The following remarks pertain to this problem.

(1) The class of strictly sofic systems is not closed under uniformly finite-to-one symbolic homomorphisms. Let (Y, σ) be the "even" system of Weiss [15]: $Y = \{y \mid y_i = 0$ or $1, 0\overbrace{1 \cdots 1}^{\text{odd}}0$ does not appear in $Y\}$. Then $Y = X(g)$ where $M = x \begin{pmatrix} 11 \\ 10 \end{pmatrix}$ and g corresponds to $\left(\begin{pmatrix} 10 \\ 00 \end{pmatrix}, \begin{pmatrix} 01 \\ 10 \end{pmatrix}\right)$. If M is indexed by 0 and 1, then $g(x_1 x_2) = x_1 + x_2 (\text{mod}\, 2)$. It is easy to see that (Y, σ) is TPPD and strictly

sofic. Let $f: \mathscr{B}(Y,3) \to \{0,1\}$ be defined by $f(001) = f(110) = 1$ and $f(B) = 0$ otherwise. Then f_∞ is uniformly finite-to-one and $f_\infty(Y) = X\begin{pmatrix}11\\10\end{pmatrix}$.

(2) The zeta function does not distinguish between subshifts of finite type and strictly sofic systems. Bowen and Lanford [6] show that the zeta function of $(X(M),\sigma)$ is given by $\zeta_{(X(M),\sigma)}(z) = [\det(I - zM)]^{-1}$. With minor modifications, the procedure used by Manning [11] to calculate the zeta function of an Axiom A diffeomorphism can be used to calculate the zeta function of $(X(f),\sigma)$ when f_∞ is uniformly finite-to-one.

Let

$$M = \begin{pmatrix}110\\001\\110\end{pmatrix}$$

and let f be the 2-block map of $(X(M),\sigma)$ corresponding to the partition

$$\left(\begin{pmatrix}100\\001\\010\end{pmatrix}, \begin{pmatrix}010\\000\\100\end{pmatrix}\right).$$

The Bowen-Lanford formula and Manning's procedure can be used to show that

$$\zeta_{(X(M),\sigma)}(z) = \zeta_{(X(f),\sigma)}(z) = 1/(1 - z - z^2).$$

It is easy to see that $(X(M),\sigma)$ is isomorphic to $(X\begin{pmatrix}11\\10\end{pmatrix},\sigma)$, and in this setting $X(f)$ is the image of $X\begin{pmatrix}11\\10\end{pmatrix}$ under g_∞, where $g(x_1x_2x_3) = x_1 + x_3 \pmod 2$. It is easily checked that $(X(f),\sigma)$ is strictly sofic.

The zeta function of a strictly sofic system, although rational, need not be the reciprocal of a polynomial. For example, the zeta function of the "even" system is $(1+z)/(1-z-z^2)$.

(3) A TPPD subshift of finite type with a fixed point must be (topologically) mixing, however, non-mixing TPPD sofic systems with fixed points are easily constructed.

(4) A "spectral decomposition" theorem (cf. [13, p. 777]) holds for subshifts of finite type: the non-wandering set of $(X(M),\sigma)$ is a disjoint union of finitely many TPPD systems, namely the $(X(M_k),\sigma)$'s. The corresponding result is not true for strictly sofic systems. Let $Y = \{y \mid y_i = 0 \text{ or } 1\} \cup \{y \mid y_i = 0 \text{ or } 2\}$.

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WESLEYAN UNIVERSITY
MIDDLETOWN, CONNECTICUT 06457 U.S.A.
AND
UNIVERSITY OF MARYLAND, BALTIMORE COUNTY
BALTIMORE, MARYLAND 21228 U.S.A.